Mexican Hat Wavelet on the Heisenberg Group

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Abstract

In this article wavelets (admissible vectors) on the Heisenberg group $\mathbb{H}$ are studied from the point of view of *Calderón*'s formula. We shall define **Calderón admissible** vectors in Definition 2.1 . Further in Theorem 2.2 we show that for the class of Schwartz functions the Calderón admissibility condition is equivalent to the usual admissibility property which will be introduced in this work.
Furthermore motivated by a well-known example on the real line, the *Mexican-Hat* wavelet, we demonstrate the existence and construction of an analogous wavelet on the Heisenberg Lie group with 2 vanishing moments, which together with all of its derivatives has "Gaussian" decay. The precise proof can be found in Theorem 3.2.

Keywords. *Wavelets, admissible vectors, Schwartz functions, Heisenberg groups, sub-Laplacian operator, Rockland operator, heat kernel.*

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1 Introduction and Definitions

The Heisenberg group $\mathbb{H}$ is a Lie group with underlying manifold $\mathbb{R}^3$. We denote points in $\mathbb{H}$ by (p, q, t) with $p, q, t \in \mathbb{R}$, and define the group operation by

$$(p_1, q_1, t_1) * (p_2, q_2, t_2) = \big(p_1 + p_2,\ q_1 + q_2,\ t_1 + t_2 + \frac{1}{2}(p_1 q_2 - q_1 p_2)\big). \tag{1}$$

It is straightforward to verify that $*$ is a group operation. We can identify both $\mathbb{H}$ and its Lie algebra $\mathfrak{h}$ with $\mathbb{R}^3$, with group operation given by (1) and Lie bracket given by

$$[(p_1, q_1, t_1), (p_2, q_2, t_2)] = (0, 0, p_1 q_2 - q_1 p_2).$$

The Haar measure on the Heisenberg group $\mathbb{H} = \mathbb{R}^3$ is the usual Lebesgue measure. More precisely, the Lie algebra $\mathfrak{h}$ of the Heisenberg group $\mathbb{H}$ has a basis $\{X, Y, T\}$, which we may think of as left invariant differential operators on $\mathbb{H}$; where $[X, Y] = T$ and all other brackets are zero, and where the exponential function $\exp :\ \mathfrak{h} \to \mathbb{H}$ is the identity, i.e.,

$$\exp\,(pX + qY + tT) = (p, q, t).$$

The action of $\mathfrak{h}$ on space $C^\infty(\mathbb{H})$ via the left invariant differential operators $\{X, Y, T\}$ is defined by the following formula:

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Suppose $f \in C^\infty(\mathbb{H})$, then

$$\begin{aligned}(Xf)(p,q,t) =& \frac{d}{dp}f(p,q,t) - \frac{1}{2}q\frac{d}{dt}f(p,q,t), \\ (Yf)(p,q,t) =& \frac{d}{dq}f(p,q,t) + \frac{1}{2}p\frac{d}{dt}f(p,q,t), \\ (Tf)(p,q,t) =& \frac{d}{dt}f(p,q,t).\end{aligned} \tag{2}$$

Our definition of continuous wavelet transform for the Heisenberg group will be from the representation point of view adapted from the case $\mathbb{R}$. For the construction of wavelet transform one needs a one-parameter group of dilations for $\mathbb{H}$. Here we consider $H := (0,\infty)$ as the one-parameter dilation group of $\mathbb{H}$ which is defined as follows: Suppose $a > 0$. Then the operator δ_a defines an automorphism of $\mathbb{H}$ by

$$\delta_a(p,q,t) = (ap, aq, a^2t) \qquad \forall (p,q,t) \in \mathbb{H}. \tag{3}$$

The set $\{\delta_a : \ a > 0\}$ forms a group of automorphisms of $\mathbb{H}$, called the dilation group for $\mathbb{H}$ (for more details about such dilation groups see for example [3]). We denote the operation of δ_a by $\delta_a(\omega) = a\omega$ for any $\omega \in \mathbb{H}$. From now on, $a > 0$ refers to the automorphism δ_a and $H = (0,\infty)$ denotes the closed subgroup of automorphisms of $\mathbb{H}$ with operation as in (3).
The group $H = (0,\infty)$ operates continuously by topological automorphisms on the locally compact group $\mathbb{H}$. So we can define the semidirect product $G := \mathbb{H} \rtimes (0,\infty)$, which is a locally compact topological group with the product topology. Elements of G can be written as $(\omega, a) \in \mathbb{H} \times (0,\infty)$ and the group operation on G is defined by

$$(\omega, a)(\acute{\omega}, \acute{a}) = (\omega(a\acute{\omega}), a\acute{a}) \quad \forall \omega, \acute{\omega} \in \mathbb{H} \text{ and } \forall a, \acute{a} > 0.$$

G is a non-unimodular group and its left Haar measure is given by $d\mu_G(\omega,a) = a^{-5}d\omega da$. Analogously to the situation on $\mathbb{R}$, for $a > 0$ the dilation operator D_a is defined by $D_af(.) = a^{-2}f(a^{-1}.)$ and for $\omega \in \mathbb{H}$, L_ω denotes the left translation where $L_\omega f(.) = f(\omega^{-1}.)$ for any f defined on $\mathbb{H}$ and $\omega \in \mathbb{H}$.

Definition 1.1 *For any $(\omega, a) \in G$ and $f \in L^2(\mathbb{H})$ define*

$$(\pi(\omega,a)f)(\upsilon) := L_\omega D_a f(\upsilon) = a^{-2}f(a^{-1}(\omega^{-1}.\upsilon)). \tag{4}$$

It is easy to prove that π is a strongly continuous unitary representation of G which acts on $L^2(\mathbb{H})$ by (4). This representation is called the "quasi-regular representation".

Next we give the definition of admissible vectors in $L^2(\mathbb{H})$, which arises from the action of the quasi-regular representation on $L^2(\mathbb{H})$ by (4).

Definition 1.2 *For any $\phi \in L^2(\mathbb{H})$ the associated coefficient operator V_ϕ is defined on $L^2(\mathbb{H})$ by*

$$V_\phi(f)(\omega,a) = \langle f, \pi(\omega,a)\phi\rangle \quad \forall f \in L^2(\mathbb{H}),\ (\omega,a) \in G.$$

ϕ is called **admissible** *if V_ϕ maps $L^2(\mathbb{H})$ into $L^2(G)$ isometrically up to a constant, i.e.,*

$$\| f \|^2 = const. \int_{\mathbb{H}} \int_0^\infty | V_\phi(f)(\omega,a) |^2 \, a^{-5} da d\omega \quad \forall f \in L^2(\mathbb{H}), \tag{5}$$

where the constant is positive and only depends on ϕ. Then V_ϕ is called a continuous wavelet transform and $V_\phi(f)$ is called the continuous wavelet transform of f.

One of the important consequence of the isometry given by formula (5) is that a function can be reconstructed from its wavelet transform by means of the *"resolution identity"*, i.e, formula (5) can be read as

$$f = const. \int_{\mathbb{H}} \int_0^\infty \langle f, \pi(\omega, a)\phi \rangle \pi(\omega, a)\phi \; a^{-5} da d\omega \quad \forall \; f \in L^2(\mathbb{H}), \tag{6}$$

which the convergence of the integral is understood in the weak sense.

The most importance of wavelet theory is its microscope effect, i.e, by choosing a suitable wavelet ϕ, as the lens, one can obtain information about the local regularity of argument functions f in $L^2(\mathbb{H})$. This information is obtained from the wavelet coefficients $\langle f, \pi(\omega, a)\phi \rangle$ when for instance these coefficients have a fast decay when $a \to 0$. Note that in the following definition we take the Schwartz functions on the Heisenberg group to be the Schwartz functions on $\mathbb{R}^3$.
The existence of admissible vectors for the quasi-regular representation of $G := N \rtimes H$ on $L^2(N)$ is already proved in Führ's book [7], where N is a homogeneous Lie group and H is a one-parameter group of dilations for N (for the definition of homogeneous groups see for example [3]). However, the existence of smooth fast-decaying wavelets was left open.
Our work establishes existence of admissible radial Schwartz vectors for the case $N = \mathbb{H}$ and $H = (0, \infty)$.
The existence of admissible vectors in closed subspaces of $L^2(\mathbb{H})$ was studied in [10]. The authors consider the unitary reducible representation U of a non-unimodular group P on $L^2(\mathbb{H})$. They decompose $L^2(\mathbb{H})$ into an infinite direct sum of the irreducible invariant closed subspaces, $\mathcal{M}_n$, under the representation U on $L^2(\mathbb{H})$. Then they show that the restriction of U to these subspaces is square-integrable, i.e, each subspace $\mathcal{M}_n$ contains at last one nonzero wavelet vector with respect to U. Furthermore the authors give a characterization of the admissibility condition in the irreducible invariant closed subspaces $\mathcal{M}_n$ in the terms of the Fourier transform. But it seems that it was not trivial for the authors to show the existence of an admissible vector for all of $L^2(\mathbb{H})$. In contrast, as we will seen soon, our work first provides a characterization of admissible functions in the Schwartz space on the Heisenberg group, and then presents an explicit example of an admissible function.

It is particularly remarkable that the representation U in [10] is unitarily equivalent to the direct sum of irreducible representations, which are all square integrable. Hence by Corollary 4.27 in [7], the representation U on $L^2(\mathbb{H})$ is square integrable, i.e, there exists an admissible vector in $L^2(\mathbb{H})$. Therefore by relying on this consequence of [7] we are aware of existence of admissible functions in $L^2(\mathbb{H})$. In this work we want to study the admissibility condition for functions in $L^2(\mathbb{H})$ with respect to the quasi-regular representation, and obtain a concrete example of a Schwartz wavelet with some nice additional properties.

As we saw above, it seems that the study of wavelets on the Heisenberg group from the representation theory point of view is the usual approach to the subject. Since that method does not easily provide an explicit example for the group, we look at wavelets through an equivalent approach, which we will discuss in the next section.

We organized the new results contained in the work as follows: In section 2 we define the "Calderón admissibility" of a function on the Heisenberg group and then in Theorem 2.2 we show that the (accepted) definition of admissibility is consistent with our usage of the word of wavelets as in Definition (1.2). This theorem provides the characterization of wavelets in the class of Schwartz functions on the Heisenberg group. In section 3 we construct an explicit example of a Schwartz wavelet with two vanishing moments, such that it and all of its derivatives have "Gaussian" decay. (We say a function F on $\mathbb{H}$ has "Gaussian" decay if for some $C, c > 0$,

$$| \; F(p,q,t) \; | \leq C \exp c(p^4 + q^4 + t^2)^{-1/2} \quad \forall (p,q,t) \in \mathbb{H}.)$$

2 Calderón Admissible Functions

As mentioned before, the existence of an admissible vector for $L^2(N)$ is proved in [7], where N is a homogeneous group, for the quasi-regular representation of $G := N \rtimes H$ on $L^2(N)$. Here H is a one-parameter group of dilations of N. The existence of such vectors for the case $N := \mathbb{R}^k$ and $H < GL(k, \mathbb{R})$ has recently been studied by different authors, for instance for $k \in \mathbb{N}$ and H as a closed subgroup of $GL(k, \mathbb{R})$ by Hartmut Führ in [5] and [6], and for the case $k = 1$ and $H := \mathbb{Z}$ by the authors in [11]. The case $N := \mathbb{H}$ and $H := \mathbb{R}$ as a one-parameter group of dilation is considered by [10]. In this section we prove the existence of Schwartz admissible vectors for the case $N := \mathbb{H}$ and $H = (0, \infty)$ by applying the following definition:

Definition 2.1 *Let $\phi \in \mathcal{S}(\mathbb{H})$ and $\int \phi = 0$. Then ϕ is called* **Calderón** *admissible if for any $0 < \varepsilon < A$ and $g \in \mathcal{S}(\mathbb{H})$*

$$g * \int_{\varepsilon}^{A} \tilde{\phi}_a * \phi_a \; a^{-1} da \to cg \quad \text{as } \varepsilon \to 0;\ A \to \infty \tag{7}$$

holds in the sense of tempered distributions where c is a nonzero constant and $\phi_a(\omega) = a^{-4}\phi(a^{-1}\omega)$.

In Lemma 2.2 below, we show that on the Schwartz space the definition of admissibility in (2.1) is equivalent to the word *admissible* in the sense of Definition (1.2):

Theorem 2.2 *Let $\phi \in \mathcal{S}(\mathbb{H})$ and $\int \phi = 0$, then ϕ is admissible if and only if ϕ is Calderón admissible.*

proof: Suppose $\phi \in \mathcal{S}(\mathbb{H})$ and $g \in \mathcal{S}(\mathbb{H})$. Then according to Definition 1.2 we have:

$$\begin{aligned}
\|V_\phi g\|_2^2 &= \int_0^\infty \int_{\mathbb{H}} |\langle g, \lambda(b) D_a \phi \rangle|^2 db a^{-5} da \\
&= \int_0^\infty \int_{\mathbb{H}} |g * D_a \widetilde{\phi}(b)|^2 db a^{-5} da \\
&= \int_0^\infty \|g * D_a \widetilde{\phi}\|^2_{L^2(\mathbb{H})} a^{-5} da \\
&= \lim_{\varepsilon \to 0,\ A \to \infty} \int_{\varepsilon}^{A} \|g * D_a \widetilde{\phi}\|^2_{L^2(\mathbb{H})} a^{-5} da \\
&= \lim_{\varepsilon \to 0,\ A \to \infty} \int_{\varepsilon}^{A} \langle g * D_a \widetilde{\phi} \ ,\ g * D_a \widetilde{\phi} \rangle a^{-5} da \\
&= \lim_{\varepsilon \to 0,\ A \to \infty} \int_{\varepsilon}^{A} \langle g \ ,\ g * D_a \widetilde{\phi} * D_a \phi \rangle a^{-5} da \\
&= \lim_{\varepsilon \to 0,\ A \to \infty} \langle g \ ,\ g * \int_{\varepsilon}^{A} D_a \widetilde{\phi} * D_a \phi a^{-5} da \rangle \\
&= \lim_{\varepsilon \to 0,\ A \to \infty} \langle g \ ,\ g * \int_{\varepsilon}^{A} \widetilde{\phi_a} * \phi_a a^{-1} da \rangle.
\end{aligned} \tag{8}$$

If $K_{\varepsilon,A} = \int_{\varepsilon}^{A} \widetilde{\phi_a} * \phi_a a^{-1} da$ then $K = \lim_{\varepsilon \to 0,\ A \to \infty} K_{\varepsilon,A}$ exists in $\mathcal{S}'(\mathbb{H})$, C^∞ away from 0 and is homogeneous of degree -4, by Theorem 1.65 in [3]. Therefore if $g \in \mathcal{S}$, $g * K_{\varepsilon,A} \to g * K$ pointwise and for some N, C

$$|\ (g * K_{\epsilon,A})(x)\ | \leq C(1 + |\ x\ |)^N \qquad \text{for all } x, \varepsilon, A.$$

Using the dominated convergence theorem in (8), if $g \in \mathcal{S}(\mathbb{H})$, then

$$\langle g \ ,\ \lim_{\varepsilon \to 0,\ A \to \infty} g * \int_{\varepsilon}^{A} \widetilde{\phi_a} * \phi_a a^{-1} da \rangle = \langle g, g * K \rangle \leq C \parallel g \parallel_2^2 \tag{9}$$

since the map $g \to g * K$ is bounded on $L^2(\mathbb{H})$. Thus V_ϕ maps $\mathcal{S}(\mathbb{H})$ to $L^2(G)$ and has a unique bounded extension to a map from $L^2(\mathbb{H})$ to $L^2(G)$. But if $g_k \to g$ in $L^2(\mathbb{H})$, surely $V_\phi g_k \to V_\phi g$ pointwise , so this extension can be none other than V_ϕ. Accordingly (9) holds for all $g \in L^2(\mathbb{H})$. We thus have

$$\begin{aligned}\| V_\phi g \|_2 = \| g \|_2 \quad \forall g \in L^2 &\Longleftrightarrow \langle g, g * K\rangle = \langle g, g\rangle \quad \forall g \in L^2\\ &\Longleftrightarrow g * K = g \quad \forall g \in L^2\\ &\Longleftrightarrow K = \delta \quad \text{up to a constant.}\end{aligned}$$

as desired. (In the second implication, we have used polarization.) This completes the proof. □

In the next Proposition we will obtain a sufficient condition for Schwartz functions to be admissible which is one of the chief tools for the proof of our main theorem.

Proposition 2.3 *Suppose $\phi, \psi \in \mathcal{S}(\mathbb{H})$, so that $\int \phi = 0$ and $\int \psi \neq 0$, and for some constants $k, c > 0$ and non-zero real number q one has $\widetilde{\phi}_{a^q} * \phi_{a^q} = -ac\frac{d}{da}\psi_{ka^q}$. Then ϕ is admissible.*

proof: Suppose $g \in \mathcal{S}(\mathbb{H})$ and $0 < \varepsilon < A < \infty$. By changing the coordinate a to a^q and by using the assumption that $\widetilde{\phi}_{a^q} * \phi_{a^q} = -ac\frac{d}{da}\psi_{ka^q}$, we can write

$$\begin{aligned}g * \int_\varepsilon^A \widetilde{\phi}_a * \phi_a \; a^{-1} da &= q \; g * \int_{\varepsilon^{1/q}}^{A^{1/q}} \widetilde{\phi}_{a^q} * \phi_{a^q} \; a^{-1} da\\ &= (qc) \; g * \int_{\varepsilon^{1/q}}^{A^{1/q}} \left(-a\frac{d}{da}\right) \psi_{ka^q} \; a^{-1} da \\ &= (qc) \; g * \int_{\varepsilon^{1/q}}^{A^{1/q}} \left(-\frac{d}{da}\psi_{ka^q}\right) da\\ &= (-qc) \left(g * (\psi_{Ak} - \psi_{\varepsilon k})\right)\\ &= (qc) \left(g * (\psi_{\varepsilon k} - \psi_{Ak})\right)\end{aligned} \tag{10}$$

Since $\int \psi \neq 0$, then from Proposition 1.20 [3] we have:

$$\lim_{\varepsilon \to 0} g * \psi_{\varepsilon k} = g \int \psi, \qquad \text{in } L^2\text{-norm.} \tag{11}$$

On the other hand one can write:

$$\|g * \psi_{Ak}\|_2 \leq \|g\|_1 \|\psi_{kA}\|_2 = (kA)^{-\frac{1}{2}} \|g\|_1 \|\psi\|_2 \tag{12}$$

which shows $g * \psi_{Ak} \to 0$ in L^2-norm as $A \to \infty$. Now applying (11) and (12) in (10)

$$g * \int_\varepsilon^A \phi_a * \tilde{\phi}_a a^{-1} da \to (qc) g \int \psi \quad as \quad \varepsilon \to 0, \; A \to \infty \quad \text{in } L^2\text{-norm,}$$

as desired. □

To present our next main result, we have to recall some basic definitions first:
Suppose $L = -(X^2 + Y^2)$ is the sub-Laplacian operator, where X and Y are the left-invariant vector fields on the Heisenberg group which have been defined earlier in (2). The heat kernel operator associated to L is the differential operator $\frac{d}{dt} + L$ on $\mathbb{H} \times \mathbb{R}$, where $\frac{d}{dt}$ is the coordinate vector field on $\mathbb{R}$ (one can consider this coordinate as the time coordinate). For the heat operator we recall here Proposition 1.68 of [3] for the Heisenberg group.

Proposition 2.4 *There exists a unique C^∞ function h on $\mathbb{H} \times (0,\infty)$, for which the following properties hold :*

1. $(\frac{d}{dt} + L)h = 0$ *on* $\mathbb{H} \times (0,\infty)$
2. $h(\omega,t) \geq 0,\ h(\omega,t) = h(\omega^{-1},t) \quad \forall(\omega,t) \in \mathbb{H} \times (0,\infty)$
3. $\int h(\omega,t)d\omega = 1$ *for* $t > 0$
4. $h(.,s) * h(.,t) = h(.,s+t) \quad \forall\, s,t > 0$
5. $r^4 h(r\omega, r^2 t) = h(\omega,t) \quad \forall\, \omega \in \mathbb{H},\ t,r > 0,$
   (*note that here $r\omega$ is understood as the result of applying the automorphism δ_r to ω, that is, $\delta_r(\omega)$*).

The solution h is known as the **heat kernel**.

Remarks 2.5

- *Note that the proposition 2.4 has been proved for the stratified groups in [3].*
- *Here the interval $(0,\infty)$ has nothing to do with the one-parameter group of dilations which has been introduced earlier. One may consider it as a time interval.*

The idea of this section is to apply Proposition 2.3 to $\phi(x) = Lh(x,1)$ to show that the function ϕ is an admissible vector. For that reason here we first need to compute the dilates of functions $h(.,1)$ and $Lh(.,1)$.

Lemma 2.6 *For any $a > 0$ and $\omega \in \mathbb{H}$ we have*

$$h(\omega,1)_a = a^2 h(\omega,a^2) \quad \text{and} \quad Lh(\omega,1)_a = a^2 Lh(\omega,a^2).$$

Proof: Suppose $a > 0$ and $\omega \in \mathbb{H}$. Applying #5 in Proposition 2.4 one finds:

$$h(\omega,1)_a = a^{-4} h(a^{-1}\omega,1) = a^2 h(\omega,a^2). \tag{13}$$

Similarly by applying #1 and #5 in Proposition 2.4 for $Lh(.,1)$ we have:

$$\begin{aligned} Lh(\omega,1)_a &= a^{-4} Lh(a^{-1}\omega,1) \\ &= -a^{-4}\frac{d}{dt} h(a^{-1}\omega,t)\big|_{t=1} \\ &= -\frac{d}{dt} h(\omega,a^2 t)\big|_{t=1} \\ &= a^2 Lh(\omega,a^2). \quad \square \end{aligned} \tag{14}$$

3 Mexican Hat Wavelet on $\mathbb{H}$

The purpose of this section is to show the *Calderón* admissibility of the Schwartz function $\phi = Lh(.,1)$, which will be stated in Theorem 3.2 as the other main result of this work. But first we make the following remark:

Remark 3.1 *On the real line $\mathbb{R}$, the heat kernel, as our motivating example, is given by $h(x,t) = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4t}}$, and $h(x,1) = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}$. The second derivative of the Gaussian is an often employed wavelet, the Mexican-Hat wavelet. This function (and its dilated and translated copies) has a shape similar to a Mexican hat (for more see for example [2]) This wavelet has two vanishing moments and evidently it and all its derivatives have Gaussian decay. Our goal in the next theorem is prove the existence of a Mexican-Hat wavelet on the Heisenberg group with similar properties.*

Theorem 3.2 *The Schwartz function $\phi(\omega) = Lh(\omega, 1)$ is admissible and it and all of its derivatives have "Gaussian" decay.*

Proof: First we shall show that $\tilde{\phi} = \phi$. This is easy to see since for any $\omega \in \mathbb{H}$ and $t > 0$ by applying #2 in Proposition 2.4, we find

$$\widetilde{Lh}(\omega, t) = -\frac{d}{dt}\tilde{h}(\omega, t) = -\frac{d}{dt}h(\omega, t) = Lh(\omega, t).$$

To prove the theorem, it is sufficient to show that for the function $\psi = h(\omega, 1) + Lh(\omega, 1)$ the relation

$$\widetilde{\phi}_{\sqrt{a}} * \phi_{\sqrt{a}} = \phi_{\sqrt{a}} * \phi_{\sqrt{a}} = -ca\frac{d}{da}\psi_{\sqrt{2a}} \tag{15}$$

holds. Hence by applying Proposition 2.3 we will get our assertion.
Using the relations (13) and (14), we find

$$\begin{aligned} \phi_{\sqrt{a}} * \phi_{\sqrt{a}} &= (Lh(.,1))_{\sqrt{a}} * (Lh(.,1))_{\sqrt{a}} \\ &= aLh(.,a) * aLh(.,a). \end{aligned} \tag{16}$$

But for any $a, b > 0$,

$$\begin{aligned} Lh(.,a) * Lh(.,b) &= \frac{d}{da}h(.,a) * \frac{d}{db}h(.,b) \\ &= \frac{d}{da}\frac{d}{db}h(.,a+b) \\ &= L^2h(.,a+b). \end{aligned}$$

If $a = b$ we get

$$\phi_{\sqrt{a}} * \phi_{\sqrt{a}} = a^2L^2h(.,2a), \tag{17}$$

while, by using (14), we find

$$\begin{aligned} \psi_{\sqrt{2a}} &= (h(.,1))_{\sqrt{2a}} + (Lh(.,1))_{\sqrt{2a}} \\ &= h(.,2a) + 2aLh(.,2a). \end{aligned}$$

Observe that the derivative of $\psi_{\sqrt{2a}}$ with respect to the parameter a is computed as follows :

$$\begin{aligned} \frac{d}{da}\psi_{\sqrt{2a}} &= \frac{d}{da}h(.,2a) + 2Lh(.,2a) + 2a\frac{d}{da}Lh(.,2a) \\ &= 2\frac{d}{d2a}h(.,2a) + 2Lh(.,2a) + 4a\frac{d}{d2a}Lh(.,2a) \\ &= -4aL^2h(.,2a). \end{aligned} \tag{18}$$

Comparing the equations (16) and (18), we see that the relation (15) holds for ϕ, ψ, and for $c = 4$, as desired.
The fact that the function ϕ has the property that it and all of its derivatives have "Gaussian" decay is known by the work of Jersion and Sanchez-Calle [9] and of Varopoulos [13] and hence we are done. □

4 Some Remarks

1. In this article, we provided our results for the Heisenberg group, $\mathbb{H} \simeq \mathbb{R}^3$, only for the sake of simplicity; evidently our main results in this work hold for the Heisenberg group $\mathbb{H}^n$ for any n also, i.e. with the underline manifold $\mathbb{R}^{2n+1}$.

2. Observe that the main results of this article can also be achieved for the general case of stratified Lie groups of any homogeneous degree, since Proposition 2.4 has been stated for this class of groups. (For more about stratified groups and homogeneous degree, we refer the interested reader to [3].)

3. Again for simplicity, here we considered only the sub-Laplacian operator L on the group, but certainly one can see that the results hold for any positive Rockland operator, which is defined for example in [3].

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